On the Dynamic Programming approach to economic models governed by DDE's

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Abstract

In this paper we consider a family of optimal control problems for economic models whose state variables are driven by Delay Differential Equations (DDE's). We consider two main examples: an AK model with vintage capital and an advertising model with delay effect. These problems are very difficult to treat for three main reasons: the presence of the DDE's, that makes them infinite dimensional; the presence of state constraints; the presence of delay in the control. Our main goal is to develop, at a first stage, the Dynamic Programming approach for this family of problems. The Dynamic Programming approach has been already used for similar problems in cases when it is possible to write explicitly the value function V (see [17]). Here we deal with cases when the explicit form of V cannot be found, as most often occurs. We carefully describe the basic setting and give some first results on the solution of the Hamilton-Jacobi-Bellman (HJB) equation as a first step to find optimal strategies in closed loop form.

1 Introduction

In this paper we want to develop the Dynamic Programming approach for a family of optimal control problems related to economic models governed by Delay Differential Equations (DDE's).

The presence of DDE's makes the problem difficult to treat. One possible way of dealing with DDE's - the one we choose - is rewriting the problem as an optimal control problem governed by ODE's in a suitable Hilbert space. Although such infinite dimensional optimal control problems have already been studied, the present literature does not cover our case, as it does not include the following features:

- the presence of unbounded operators coming from the DDE which is not analytic and does not satisfy smoothing assumptions;
- the presence of state/control constraints (which is indeed peculiar of economic models);

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- the fact that the delay appears in the state and in the control (causing the control operator to be possibly unbounded).

We stress the fact that these difficulties are the rule in economic models governed by DDE's.

Here we consider problems with linear DDE's and *concave* objective functional: concavity will play a key role in the paper. When concavity lacks, one can still apply Dynamic Programming in the framework of viscosity solutions - which we avoid here. Nevertheless, we address the reader to [8] for a standard reference on viscosity solutions.

We remark that this is a first step in treating such kind of problems. We already studied thoroughly in [17] a case where explicit solution of the associated Hamilton-Jacobi-Bellman (HJB) equation can be found (in such case the problem is much easier to treat). Here we want to develop the Dynamic Programming approach in those cases when explicit solutions of the associated HJB equation are not available. We here develop the finite horizon case. The infinite horizon case can be treated with our method using a limiting procedure when the horizon goes to $+\infty$ but we leave it for future work[1].

The main result of the paper is that the value function of the problem is a solution, in a suitable weak sense, of the HJB equation. This a first step towards the so-called Verification Theorem which is a powerful tool to study the optimal paths of the problem and which is the subject of our current research.

We concentrate on two main examples: an AK model with vintage capital, taken from [7] (see also [6] and [17]) and an advertising model with delay effects (see [24, 25]) that are exposed in Section 2.

The plan of the paper is the following. In Section 2 we present the applied examples. In Section 3 we recall the basic steps of the Dynamic Programming approach and we give an overview of the current literature on the Dynamic Programming for infinite dimensional optimal control problems. In Section 4 we rewrite the state equation of such problems as an ODE in a suitable Hilbert space, concentrating on the first example, as the second can be rephrased similarly. In Section 5 we write the resulting infinite dimensional optimal control problem and its HJB equation. Section 6 we show our main result: the existence of an ultraweak solution of the HJB equation. The Appendix 7 contains some definition and proof that may be useful for the reader.

2 Two examples

We present the two applied problems motivating this paper.

2.1 An AK model with vintage capital

We consider here an optimal control problem related to a generalization of the model presented by Boucekkine, Puch, Licandro and Del Rio in [7]. Indeed, we assume that the system is ruled by the same evolution law as the one in [7], that is, an AK growth model with a stratification on the capital. Besides, we consider the finite horizon problem with a (more) general concave target functional, as

[1] In thus respect we can say that the finite horizon case is as a first step towards the infinite horizon one.

specified later. The analysis of such a model proves interesting in the study of short run fluctuations and of transitional dynamics: the reader is referred to [7] for a deep discussion upon this and other related matters. The model of [7] is an infinite horizon model, while here we consider the finite horizon case. As mentioned in the introduction, this is a first step towards the infinite horizon case.

The AK-growth model with vintage capital is based on the following accumulation law for capital goods

$$k(s) = \int_{s-R}^{s} i(\sigma)\mathrm{d}\sigma$$

where $i(\sigma)$ is the investment at time σ. That is, capital goods are accumulated for the length of time R (scrapping time) and then dismissed. Note that such an approach introduces a differentiation in investments that depends on their age. If we assume a linear production function, that is

$$y(s) = ak(s)$$

where $y(s)$ is the output at time s (note that "AK" reminds of the linear dependence of the dynamic from the trajectory - a constant A multiplied by K; such constant A is a in our case), and we assume also the accounting relation

$$y(s) = c(s) + i(s),$$

meaning that at every time the social planner chooses how to split the production into consumption $c(s)$ and investment $i(s)$, then the state equation may be written into infinitesimal terms as follows

$$\dot{k}(s) = ak(s) - ak(s-R) - c(s) + c(s-R), \quad s \in [t,T]$$

i.e. as a DDE. The time variable s varies in $[t,T]$, with t the *initial time* and T the (finite) *horizon* of the problem. Indeed, the social planner has to maximize the following functional

$$\int_t^T e^{-\rho s} h_0(c(s))\mathrm{d}s + \phi_0(k(T)) \tag{1}$$

where h_0 and ϕ_0 are concave u.s.c. utility functions. We recall that in [7] the horizon is infinite and $\phi_0 = 0$. Moreover the instantaneous utility is CRRA (i.e. Costant Relative Risk Aversion), that is the function h_0 is of type $h_0(c) = \frac{c^{1-\sigma}}{1-\sigma}$, which satisfies our assumptions as a subcase.

Observe that we take the starting time t to be variable to apply the finite horizon dynamic programming.

We assume that the capital at time s (and consequently the production) and the consumption at time s cannot be negative:

$$k(s) \geq 0, \qquad c(s) \geq 0, \quad \forall s \in [t,T] \tag{2}$$

These constraints are different from the more restrictive and more natural ones of [7], where also the investment path $i(\cdot)$ was assumed positive.

The main reason for such a choice is technical: we cannot apply the strong solution approach that we use in this work with mixed constraints such as

those in [7]. The treatment of mixed constraints is also left for future work. We mention indeed that the optimal solutions for the problem without mixed constraints may satisfy in some cases the positivity of investments, yielding the solution also for the problem with mixed constraints.

In order to take the constraints into account, we assume that the consumption (that is, the control variable of the system) lies in the following admissible set

$$\mathcal{A} \stackrel{def}{=} \{c(\cdot) \in L^2([t,T],\mathbb{R}) \; : \; c(\cdot) \geq 0 \; and \; k(\cdot) \geq 0 \; a.e. \; in \; [t,T]\}.$$

2.2 An advertising model with delay effects

Another example of optimal control problems driven by DDE's is the following a dynamic advertising model presented in the stochastic case in the papers [25, 24], and, in deterministic one, in [21] (see also [?] and the references therein for related models)[2].

Let $t \geq 0$ be an initial time, and $T > t$ a terminal time ($T < +\infty$ here). Moreover let $\gamma(s)$, with $0 \leq t \leq s \leq T$, represent the stock of advertising goodwill of the product to be launched. Then the general model for the dynamics is given by the following controlled Delay Differential Equation (DDE) with delay $R > 0$ where z models the intensity of advertising spending:

$$\begin{cases} \dot{\gamma}(s) = a_0\gamma(s) + \int_{-R}^{0} \gamma(s+\xi)da_1(\xi) + b_0 z(s) + \int_{-R}^{0} z(s+\xi)db_1(\xi) \quad s \in [t,T] \\ \\ \gamma(t) = x; \quad \gamma(\xi) = \theta(\xi), \; z(\xi) = \delta(\xi) \;\; \forall \xi \in [t-R,t], \end{cases} \tag{3}$$

with the following assumptions:

- a_0 is a constant factor of image deterioration in absence of advertising, $a_0 \leq 0$;
- $a_1(\cdot)$ is the distribution of the forgetting time, $a_1(\cdot) \in L^2([-R,0];\mathbb{R})$;
- b_0 is a constant advertising effectiveness factor, $b_0 \geq 0$;
- $b_1(\cdot)$ is the density function of the time lag between the advertising expenditure z and the corresponding effect on the goodwill level, $b_1(\cdot) \in L^2([-R,0];\mathbb{R}_+)$;
- x is the level of goodwill at the beginning of the advertising campaign, $x \geq 0$;
- $\theta(\cdot)$ and $\delta(\cdot)$ are respectively the goodwill and the spending rate before the beginning, $\theta(\cdot) \geq 0$, with $\theta(0) = x$, and $\delta(\cdot) \geq 0$.

Note that when $a_1(\cdot)$, $b_1(\cdot)$ are identically zero, equation (3) reduces to the classical model contained in the paper by Nerlove and Arrow (1962). We assume that the goodwill and the investment in advertising at each time s cannot be negative:

$$\gamma(s) \geq 0, \qquad z(s) \geq 0, \quad \forall s \in [t,T]. \tag{4}$$

[2] We observe that also other models of delay type arising in economic theory can be treated with our tools (see *e.g.* the paper by [6]).

Finally, we define the objective functional as

$$J(t,x;z(\cdot)) = \varphi_0(\gamma(T)) - \int_t^T h_0(z(s))\,ds, \tag{5}$$

where φ_0 is a concave utility function, h_0 is a convex cost function, and the dynamic of γ is determined by (3). The functional J has to be maximized over some set of admissible controls $\mathcal{U}$, for instance $L^2([t,T];\mathbb{R}_+)$, the space of square integrable nonnegative functions.

3 The dynamic programming approach

The Dynamic Programming (DP) approach to optimal control problems can be summarized in four main steps (see for instance Fleming and Rishel [23] for the DP in the finite dimensional case and Li and Yong [29] for the DP in the infinite dimensional case):

(i) letting the initial data vary, calling *value function* the supremum of the objective functional and writing an equation whose candidate solution is the value function: the so-called DP Principle, together with its infinitesimal version, the Hamilton-Jacobi-Bellman (HJB) equation;

(ii) solving (whenever possible) the HJB equation to find the value function;

(iii) proving that the present value of the optimal control strategy can be expressed as a function of the present value of the optimal state trajectory: a so-called closed loop (or feedback) relation for the optimal control;

(iv) solving, if possible, the Closed Loop Equation (CLE), i.e. the state equation where the control is replaced by the closed loop relation: the solution is the optimal state trajectory and the optimal control strategy is consequently derived from the closed loop relation.

Such method, when applicable, allows one to give a powerful description of the optimal paths of an optimal control problem.

First of all we clarify that the two models above are not easy to manage with the DP approach as they presents two special difficulties.

- The state equation is a Delay Differential Equation while the DP approach is generally formulated for controlled Ordinary Differential Equation (ODE). One way to approach the issue (for a different one, see e.g. Kolmanowskii and Shaikhet [28]) is to rewrite the DDE as an ODE in an infinite dimensional space, which plays the role of the state space. We use in the sequel the techniques developed by Delfour, Vinter and Kwong (see Section 4 below for explanation and Subsection 3.1 for references). It must be noted that the resulting infinite dimensional control problem is harder than the ones usually treated in the literature (see e.g. [29]) due to the unboundedness of the control operator and the non-analyticity of the semigroup involved (see again Subsection 4).
- Both problem feature pointwise constraints on the state variable, see (2), (4). Their presence makes the problem much more difficult, and only a few

results in special cases (different from the one treated here) are available in the literature. Indeed for such problems in infinite dimension there is no well established theory. This fact is at the basis of the theoretical problem contained in the paper [7] and mentioned in [17] point (II) in the introduction: show that the candidate optimal trajectory satisfies the pointwise constraints (2).

To overcome such difficulties in [17] we show that for our special problem we can exhibit an explicit solution of HJB equation. This is the key result that allows to complete the DP approach in [17].

Here, since we do not want to write the utility functions in a fixed explicit form (like the CRRA used in [7, 17]), we cannot obtain an an explicit solution of HJB equation. Therefore we would like (here and in the future) to perform the following steps: proving existence (and possibly, uniqueness) for the HJB equation, then some theoretical results of type (iii) and (iv) above, and hopefully some subsequent numerical approximation. This is a wide and difficult program. In this paper we take just a first step towards the scope: existence results for the HJB equation.

3.1 The literature on Delay Differential Equations and on Dynamic Programming in infinite dimensions

For Delay Differential Equations a recent, interesting and accurate reference is the book by Diekmann, van Gils, Verduyn, Lunel and Walther [16].

The idea of writing delay system using a Hilbert space setting was first due to Delfour and Mitter [14], [15]. Variants and improvements were proposed by Delfour [11], [9], [10], Vinter and Kwong [30], Delfour and Manitius [12], Ichikawa [26] (see also the precise systematization of the argument in chapter 4 of Bensoussan, Da Prato, Delfour and Mitter [5]).

The optimal control problem in the (linear) quadratic case is studied in Vinter, Kwong [30], Ichikawa [27], Delfour, McCalla and Mitter [13]. In that case the Hamilton-Jacobi-Bellman reduces to the Riccati equation.

The study of Hamilton-Jacobi-Bellman equation in Hilbert spaces, started with the papers of Barbu and Da Prato [1], [2], [3], is a large and diversified research field. We recall that the best one may achieve is a "classical" solution of HJB equations (i.e. solutions that are differentiable in time and state) since this allows to get a more handleable closed loop form of the optimal strategy. Since classical solutions are not always available, there is a second stream in the literature that studies the existence of "weak" solutions (i.e. solutions that are not differentiable)[3]. In this paper we investigate existence of a weak-type solution (that we call *ultraweak*, see Section 6) that are limits of classical solutions. Up to now, to our knowledge, the existence of such solutions for the HJB equation in cases where the state equation is a Delay Differential Equation has not been studied in the literature (apart from the linear quadratic case). In the economic literature the study of infinite dimensional optimal control problems that deals with vintage/heterogeneous capital or advertising models is a quite recent tool but of growing interest: see for instance [4], [22], [18], [24, 25].

[3] The most general concept of weak solution is the one of viscosity solution, introduced by Crandall and Lions in the finite dimensional case and then applied to infinite dimension by the same authors, see [8] for an introduction to the topic and further references.

4 The state equation in an infinite dimensional setting.

In this section we show how to rewrite the state equations of our examples as controlled ODE's in a suitable Hilbert space. We do it thoroughly for the first example, as the second is similar and simpler.

4.1 Notation and preliminary results

In this section we recall some general results on delay differential equations (DDE) and on the related Hilbert space approach, as applied to our case. The reader is referred to the book by Bensoussan, Da Prato, Delfour and Mitter [5] for details. We consider from now on fixed $R > 0$, and $a > 0$. With notation similar to that of [5], given $T > t \geq 0$ and $z \in L^2([t-R,T],\mathbb{R})$ (or $z \in L^2_{loc}([t-R,+\infty),\mathbb{R})$), for every $s \in [t,T]$ (or $s \in [t,+\infty)$) we call $z_s \in L^2([-R,0];\mathbb{R})$ the function

$$\begin{cases} z_s \colon [-R,0] \to \mathbb{R} \\ z_s(\sigma) \stackrel{def}{=} z(s+\sigma) \end{cases}$$

Given a control $c \in \mathcal{A}$ we consider the the following delay differential equation:

$$\begin{cases} \dot{k}(s) = ak(s) - ak(s-R) - c(s) + c(s-R) \quad for\ s \in [t,T] \\ (k(t), k_t, c_t) = (\phi^0, \phi^1, \omega) \in \mathbb{R} \times L^2([-R,0];\mathbb{R}) \times L^2([-R,0];\mathbb{R}) \end{cases} \tag{6}$$

where k_t and c_t are interpreted by means of the definition above. Note that in the delay setting the initial data are a triple, whose first component is the state, the second and third are respectively the history of the state and the history of the control up to time t (more precisely, on the interval $[t-R,t]$). The equation does not make sense pointwise, but has to be regarded in integral sense. We give now a more precise existence result and an estimate on the solution:

Theorem 4.1. *Given an initial condition $(\phi^0, \phi^1, \omega) \in \mathbb{R} \times L^2([-R,0];\mathbb{R}) \times L^2([-R,0];\mathbb{R})$ and a control $c \in L^2([t,T],\mathbb{R})$ there exists a unique solution $k(\cdot)$ of (6) in $W^{1,2}([t,T],\mathbb{R})$. Moreover there exists a positive constant $C(T-t)$ such that*

$$|k|_{W^{1,2}([t,T],\mathbb{R})} \leq C(T-t)\Big(|\phi^0| + |\phi^1|_{L^2([-R,0];\mathbb{R})} + |\omega|_{L^2([-R,0];\mathbb{R})} + |c|_{L^2([t,T],\mathbb{R})}\Big) \tag{7}$$

Proof. See [5] Theorem 3.3, p.217 for the first part and Theorem 3.3 p.217, Theorem 4.1 p.222 and p.255 for the second statement. □

In view of the continuous embedding $W^{1,2}([t,T],\mathbb{R}) \hookrightarrow C^0([t,T],\mathbb{R})$ we have also:

Corollary 4.2. *There exists a positive constant (possibly different from the one above) $C(T-t)$ such that*

$$|k|_{C^0([t,T],\mathbb{R})} \leq C(T-t)\Big(|\phi^0| + |\phi^1|_{L^2([-R,0];\mathbb{R})} + |\omega|_{L^2([-R,0];\mathbb{R})} + |c|_{L^2([t,T],\mathbb{R})}\Big) \tag{8}$$

We consider now the continuous linear application L with norm $\|L\|$

$$
\begin{aligned}
&L\colon C([-R,0],\mathbb{R}) \to \mathbb{R}\\
&L\colon \varphi \mapsto \varphi(0) - \varphi(-R)
\end{aligned}
$$

and then define $\mathcal{L}^t$ as follows

$$
\begin{aligned}
&\mathcal{L}^t\colon C_c([t-R,T],\mathbb{R}) \to L^2([t,T],\mathbb{R})\\
&\textit{where}\ \ \mathcal{L}^t(\phi)\colon s \mapsto L(\phi_s) \quad \textit{for}\ s\in[t,T]
\end{aligned}
\tag{9}
$$

where $C_c(t-R,T;\mathbb{R})$ is the set of real continuous functions having compact support contained in $(t-R,T)$

Theorem 4.3. *The linear operator* $\mathcal{L}^t\colon C_c([t-R,T],\mathbb{R}) \to L^2([t,T],\mathbb{R})$ *has a continuous extension* $\mathcal{L}^t\colon L^2([t-R,T],\mathbb{R}) \to L^2([t,T],\mathbb{R})$ *with norm* $\leq \|L\|$ *.*

Proof. See [5] Theorem 3.3, p. 217. □

Using the "L" notation we can rewrite (6) as

$$
\begin{cases}
\dot{k}(s) = aL(k_s) - L(c_s) \quad \textit{for}\ s\in[t,T]\\
(k(t),k_t,c_t) = (\phi^0,\phi^1,\omega) \in \mathbb{R}\times L^2([-R,0];\mathbb{R})\times L^2([-R,0];\mathbb{R})
\end{cases}
$$

and using the "$\mathcal{L}^t$" notation we can rewrite (6) as

$$
\begin{cases}
\dot{k}(s) = a(\mathcal{L}^t k)(s) - (\mathcal{L}^t c)(s) \quad \textit{for}\ s\in[t,T]\\
(k(t),k_t,c_t) = (\phi^0,\phi^1,\omega) \in \mathbb{R}\times L^2([-R,0];\mathbb{R})\times L^2([-R,0];\mathbb{R})
\end{cases}
\tag{10}
$$

There follows another step towards the setting in infinite dimension that we intend to use. So far, the history of the control and of the trajectory were kept separated. Indeed one may note that the delay system depends jointly on those data. Such joint dependence is exploited in the sequel to reduce the dimension of the state space. We then need to add some more notation to make this more explicit.

- Given $u\in L^2([t-R,T],\mathbb{R})$ we define the function $e^t_+u \in L^2([t-R,T],\mathbb{R})$ as follows

$$
e^t_+u\colon [t-R,T]\to\mathbb{R}, \quad e^t_+u(s) = \begin{cases} u(s) & s\in[t,T]\\ 0 & s\in[t-R,t) \end{cases}
$$

- Given $u\in L^2([-R,0];\mathbb{R})$ we define the function $e^0_-u \in L^2([t-R,T],\mathbb{R})$ as follows

$$
e^0_-u\colon [t-R,T]\to\mathbb{R}, \quad e^0_-u(s) = \begin{cases} 0 & s\in[t,T]\\ u(s-t) & s\in[t-R,t) \end{cases}
$$

- Given a function $u\in L^2([-R,0];\mathbb{R})$ and $s\in[t,T]$ we define the function $\eta(s)u \in L^2([-R,0];\mathbb{R})$ as follows

$$
\eta(s)u\colon [-R,0]\to\mathbb{R}, \quad (\eta(s)u)(\theta) = \begin{cases} u(-s+t+\theta) & \theta\geq -R+s-t\\ 0 & \theta < -R+s-t \end{cases}
$$

Note that $k = e^t_+ k + e^0_- \phi^1$, and $c = e^t_+ c + e^0_- \omega$, then we can separate the solution $k(s)$, $s \geq t$ and the control $c(s)$, $s \geq t$ from initial data ϕ^1 and ω:

$$\begin{cases} \dot{k} = a\mathcal{L}^t e^t_+ k - \mathcal{L}^t e^t_+ c + a\mathcal{L}^t e^0_- \phi^1 - \mathcal{L}^t e^0_- \omega \\ k(t) = \phi^0 \in \mathbb{R} \end{cases} \tag{11}$$

Note that system (11) does not directly use the initial function ϕ^1 and ω but only the sum of their images $a\mathcal{L}^t e^0_+ \phi^1 - \mathcal{L}^t e^0_- \omega$. We need a last step before we can write the delay equation in Hilbert space. We introduce the operator

$$\begin{cases} \overline{L}\colon L^2([-R,0];\mathbb{R}) \to L^2([-R,0];\mathbb{R}) \\ (\overline{L}\phi^1)(\alpha) \stackrel{def}{=} L(est(\phi^1)_{-\alpha})) \qquad \alpha \in (-R,0) \end{cases} \tag{12}$$

where $est(\phi^1)$ is the function $\mathbb{R} \to \mathbb{R}$ that achieves value 0 out of $(-R,0)$ and that is equal to ϕ^1 in $(-R,0)$ (the same for ω).

Note that the operator $\overline{L}$ is continuous (see [5] page 235), moreover

$$a\mathcal{L}^t e^0_- \phi^1(s) - \mathcal{L}^t e^0_- \omega(s) = (\eta(s)(a\overline{L}\phi^1 - \overline{L}\omega))(0) \qquad for\ s \geq t.$$

Therefore, if we set

$$x^1 \stackrel{def}{=} (a\overline{L}\phi^1 - \overline{L}\omega), \quad x^0 \stackrel{def}{=} \phi^0, \tag{13}$$

then we can rewrite (11) and consequently (6) as

$$\begin{cases} \dot{k}(s) = (a\mathcal{L}^t e^t_+ k)(s) - (\mathcal{L}^t e^t_+ c)(s) + (\eta(s)x^1)(0) \qquad for\ s \geq t \\ k(t) = x^0 \in \mathbb{R} \end{cases} \tag{14}$$

where $\mathbb{R} \times L^2([-R,0];\mathbb{R}) \ni x \stackrel{def}{=} (x^0, x^1)$, $c \in \mathcal{A}$. Note that (14) is meaningful for all $x \in \mathbb{R} \times L^2([-R,0];\mathbb{R})$, also when x^1 is not of the form (13). So we have embedded the original system (6) in a family of systems of the form (14).

4.2 The state equation of the AK model in the Hilbert setting

We now work on the following Hilbert space

$$M^2 \stackrel{def}{=} \mathbb{R} \times L^2([-R,0];\mathbb{R})$$

where the scalar product between two elements $\phi = (\phi^0, \phi^1)$ and $\xi = (\xi^0, \xi^1)$ is given by

$$\langle \phi, \xi \rangle_{M^2} \stackrel{def}{=} \left\langle \phi^1, \xi^1 \right\rangle_{L^2} + \phi^0 \xi^0.$$

Next we consider the homogeneous system

$$\begin{cases} \dot{z}(s) = (a\mathcal{L}^0 z)(s) \\ (z(0), z_0) = \phi \in M^2 \end{cases}$$

and define the family of continuous linear transformations on M^2

$$\begin{cases} S(s)\colon M^2 \to M^2 \\ \phi \mapsto S(s)\phi \stackrel{def}{=} (z(s), z_s). \end{cases}$$

Then $\{S(s)\}_{s\geq 0}$ is a C_0 semigroup on M^2 whose generator is

$$\begin{cases} D(G) = \left\{(\phi^0, \phi^1) \in M^2 \ : \ \phi^1 \in W^{1,2}(-R, 0) \ and \ \phi^0 = \phi^1(0)\right\} \\ G(\phi^0, \phi^1) = (aL\phi^1, D\phi^1) \end{cases}$$

where $D\phi^1$ is the first derivative of ϕ^1. A proof of this assertion can be found in [5], Chapter 4.

Note that the second component ϕ^1 of the elements of $D(G)$ is in $C([-R, 0], \mathbb{R})$ so, with a slight abuse of notation, we can re-define L on $D(G)$ in the following way

$$\begin{cases} L\colon D(G) \to \mathbb{R} \\ L(\phi^0, \phi^1) = L\phi^1 \end{cases}$$

Moreover, if $D(G)$ is endowed with the graph norm, we denote with j the continuous inclusion $D(G) \hookrightarrow M^2$. Hence the operators G, and j are continuous from $D(G)$ into M^2 and L is continuous from $D(G)$ into $\mathbb{R}$. We call G^*, j^* and L^* their adjoints, and identify M^2 and $\mathbb{R}$ with their dual spaces, so that

$$\begin{aligned} &G^*\colon M^2 \to D(G)' \\ &j^*\colon M^2 \to D(G)' \\ &L^*\colon \mathbb{R} \to D(G)' \end{aligned}$$

are linear continuous.

Definition 4.4. *The* structural state $x(s)$ *at time* $t \geq 0$ *is defined by*

$$y(s) \stackrel{def}{=} (y^0(s), y^1(s)) \stackrel{def}{=} (k(s), a\overline{L}(e^t_+ k)_s - \overline{L}(e^t_+ c)_s + \eta(s)x^1) \qquad (15)$$

In the sequel we use y^0 and y^1 to indicate respectively the first and the second component of the structural state. We can give also a different, more explicit, definition: if we call $\overleftarrow{k}_s, \overleftarrow{c}_s \in L^2([-R, 0]; \mathbb{R})$ the applications

$$\begin{aligned} &\overleftarrow{k}_s\colon \theta \mapsto -k(s - R - \theta) \\ &\overleftarrow{c}_s\colon \theta \mapsto -c(s - R - \theta) \end{aligned}$$

the structural state can be written as

$$y(s) \stackrel{def}{=} (k(s), a\overleftarrow{k}_s - \overleftarrow{c}_s + \eta(s)x^1). \qquad (16)$$

Eventually, we write the delay equation in the Hilbert space M^2 by means of the following theorem.

Theorem 4.5. *Let $y^0(s)$ be the solution of system (14) for $x \in M^2$, $c \in \mathcal{A}$ and let $y(t)$ be the structural defined in (15). Then for each $T > 0$, the state y is the unique solution in*

$$\left\{ f \in C([t, T], M^2) \ : \ \frac{\mathrm{d}}{\mathrm{d}s} j^* f \in L^2([t, T], D(G)') \right\}$$

to the following equation

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s} y(s) = G^* y(s) + L^* c(s) \\ y(t) = x. \end{cases} \qquad (17)$$

Proof. See [5] Theorem 5.1 Chapter 4. ◻

4.3 The state equation of the advertising model in the Hilbert setting

Similar arguments can be used for the advertising model. We write here only the results. We call N, B the continuous linear functionals given by

$$\begin{aligned} &N\colon C([-R,0]) \to \mathbb{R} \\ &N\colon \varphi \mapsto a_0\varphi(0) + \textstyle\int_{-r}^{0} \varphi(\xi) da_1(\xi) \end{aligned}$$

$$\begin{aligned} &B\colon C([-R,0]) \to \mathbb{R} \\ &B\colon \varphi \mapsto b_0\varphi(0) + \textstyle\int_{-r}^{0} \varphi(\xi) db_1(\xi) \end{aligned}$$

Let G be the generator of C_0-semigroup defined as:

$$\begin{cases} D(G) = \left\{ (\phi^0, \phi^1) \in M^2 \ : \ \phi^1 \in W^{1,2}(-R,0) \ and \ \phi^0 = \phi^1(0) \right\} \\ G(\phi^0, \phi^1) = (N\phi^1, D\phi^1) \end{cases}$$

We define $\overline{N}$ and $\overline{B}$ in the same way we defined $\overline{L}$ in equation (12). So we can write the advertising model in infinite dimensional form. We obtain:

- The structural state in the advertising model will have the following expression:

$$y(t) = (y^0(s), y^1(s)) \stackrel{def}{=} (\gamma(s), \overline{N}(e_+^0\gamma)_s - \overline{B}(e_+^0 z)_s + \eta(s)x^1)$$

  where $x_1 = \overline{N}(\theta) - \overline{B}(\delta)$.

- The state equation becomes

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s} y(s) = G^* y(s) + B^* z(s) \\ y(t) = x. \end{cases}$$

5 The target functional and the HJB equation

We now rewrite the profit functional for the first example in abstract terms, noting that a similar reformulation holds for the target functional of the second example. We consider a control system governed by the linear equation described in Theorem 4.5. We assume that the set of admissible controls is defined by

$$\mathcal{A} \stackrel{def}{=} \{c(\cdot) \in L^2([t,T], \mathbb{R}) \ : \ c(\cdot) \geq 0 \ and \ y^0(\cdot) \geq 0\}$$

As usual, the trajectory $y(\cdot)$ (and then $y^0(\cdot)$) depends on the choice of the control $c(\cdot)$, and of initial time and state, i.e. $y(\cdot) = y(\cdot; t, x, c(\cdot))$, but we write it explicitly only when needed.

In order to apply the results contained in [20] and recalled in the Appendix, we reformulate the maximization problem as a minimization problem. At the same time we take the constraints into account by modifying the target functional as follows. If h_0 and ϕ_0 are the concave *u.s.c.* functions appearing in (1), then we define

$$\begin{aligned} &h\colon \mathbb{R} \to \overline{\mathbb{R}} \\ &h(c) = \begin{cases} -h_0(c) & if \ c \geq 0 \\ +\infty & if \ c < 0 \end{cases} \end{aligned}$$

$$\phi\colon \mathbb{R} \to \overline{\mathbb{R}}$$
$$\phi(r) = \begin{cases} -\phi_0(r) & if\ r \geq 0 \\ +\infty & if\ r < 0 \end{cases}$$

Moreover we set

$$g\colon \mathbb{R} \to \overline{\mathbb{R}}$$
$$g(r) = \begin{cases} 0 & if\ r \geq 0 \\ +\infty & if\ r < 0 \end{cases}$$

Both h, ϕ and g are convex *l.s.c.* functions on $\mathbb{R}$. Then we define the *target functional* as

$$J(t, x, c(\cdot)) = \int_t^T e^{-\rho s}[h(c(s)) + g(y^0(s))]\mathrm{d}s + \phi(y^0(T))$$

with c varying in the *set of admissible controls* $L^2([t,T],\mathbb{R})$. It is easy to check that the problem of maximizing (1) in the class $\mathcal{A}$ is equivalent to that of minimizing J on the whole space $L^2([t,T],\mathbb{R})$. Then the original maximization problem for the AK-model has been reformulated as the following *abstract minimization problem*:

$$\inf\{J(t, x, c(\cdot))\ :\ c \in L^2([t,T],\mathbb{R}),\ \text{and } y \text{ satisfies (17)}\}, \tag{18}$$

Moreover, HJB equation is naturally associated to such minimization problem by DP, and it is given by

$$\begin{cases} \partial_t v(t,x) + \langle \nabla v(t,x), G^* x\rangle - F(t, \nabla v(t,x)) + e^{-\rho t} g(x) = 0 \\ v(T,x) = \phi_0(x) \end{cases} \tag{HJB}$$

with F defined as follows

$$\begin{cases} F\colon [0,T] \times D(G) \to \mathbb{R} \\ F(t,p) \stackrel{def}{=} \sup_{c\geq 0}\{-L(p)c - e^{-\rho t} h_0(c)\} = e^{-\rho t} h^*(-e^{\rho t} L(p)) \end{cases}$$

where h^* is the Legendre transform of the convex function h. We refer to F as to the *Hamiltonian* of the system[4].

The abstract framework is then set, and we are ready to perform Dynamic Programming.

6 The value function as ultraweak solution of HJB

We define the value function of the optimal control problem described in the previous sections as

$$W(t,x) \stackrel{def}{=} \inf_{c(\cdot)\in L^2([t,T];\mathbb{R})} J(t,x,c(\cdot)).$$

Our objective here is to provide a suitable concept of solution of HJB, so that the value function V is a solution, in such sense.

[4]Note that, following the usual definition, the Hamiltonian should be indeed $\langle p, G^* x\rangle - F(t,p) + e^{-\rho t} g(x)$. Here, for commodity of notation, we put aside of the Hamiltonian the terms which are linear or constant in p.

We recall that in [20] it is shown that, if the data satisfy certain assumptions (involving convexity, semicontinuity, and coercivity of h), then the value function of an optimal control problem with state constraints of type (18) is indeed the unique *weak* solution to a HJB equation of type (HJB), as there proved and here recalled in the Appendix, Theorem 7.11. Note that some coercivity for the function h is indeed lacking in our case, as the prototype of h_0 is $\frac{c^{1-\sigma}}{1-\sigma}$ as mentioned before, which is sublinear on the positive real axis. This causes the Hamiltonian of the problem - that is related to the Legendre transform of h_0 - to be possibly nonregular, so that all previous definition of solutions do not apply. (Note indeed that, as more precisely stated in the Appendix, a weak solution is limit of strong solutions of approximating equations, while a strong solution is itself limit of classical solutions of approximating equations. All of these notions require the Hamiltonian to be differentiable with respect to the co-state variable p.)

Here we are about to define a *ultraweak* solution as limit of weak solutions to (HJB). Note that the concept of solution is indeed generalized, although not in the same direction as before, due to the presence of possibly nonregular Hamiltonians.

According to the notation in [19], if X and Y are Banach spaces, we set

$$Lip(X;Y) = \{f : X \to Y \;:\; [f]_L := \sup_{x,y\in X,\ x\neq y} \frac{|f(x)-f(y)|_Y}{|x-y|_X} < +\infty\}$$

$$C^1_{Lip}(X) := \{f \in C^1(X) \;:\; [f']_L < +\infty\}$$

$$C_p(X,Y) := \{f : X \to \mathbb{R} \;:\; |f|_{C_p} := \sup_{x\in X} \frac{|f(x)|_Y}{1+|x|^p_X} < +\infty\}, \quad C_p(X) := C_p(X,\mathbb{R}).$$

Moreover we set

$$\Sigma_0(X) := \{w \in C_2(X) \;:\; w \text{ is convex, } w \in C^1_{Lip}(X)\}$$

$$\mathcal{Y}([0,T]\times X) = \{w : [0,T]\times X \to \mathbb{R} \;:\; w \in C([0,T],C_2(X)),\ w(t,\cdot) \in \Sigma_0(X),\ \nabla w \in C([0,T],C_1(X,X'))\}.$$

Definition 6.1. *We say that a function V is a ultraweak solution to*

$$\begin{cases} \partial_t v(t,x) + \langle \nabla v(t,x), G^* x\rangle - F(t,\nabla v(t,x)) + e^{-\rho t} g(x) = 0 \\ v(T,x) = \phi_0(x) \end{cases}$$

if there exists a sequence $\{F_n\}_n$ of functions in the space $\mathcal{Y}([0,T]\times D(G))$, such that $F_n \uparrow F$ pointwise, and

$$V(t,x) = \lim_{n\to+\infty} V_n(t,x) = \inf_{n\geq 0} V_n(t,x)$$

with V_n the unique weak solutions to

$$\begin{cases} \partial_t v(t,x) + \langle \nabla v(t,x), G^* x\rangle - F_n(t,\nabla v(t,x)) + e^{-\rho t} g(x) = 0 \\ v(T,x) = \phi_0(x) \end{cases}$$

Note that any weak solution V is convex in the state variable x, but not necessarily $l.s.c$ in (t,x). We are able to prove an existence result for equation (HJB) by proving that the value function of the control problem set in the previous section is an ultraweak solution.

Theorem 6.2. *The value function W of the optimal control problem (18) is an ultraweak solution of (HJB).*

Proof. First of all we need to construct a sequence of Hamiltonians F_n having the properties required by the definition above. We choose

$$F_n(t,p) := e^{-\rho t} h_n^*(-e^{\rho t} L(p))$$

with

$$h_n(c) = h(c) + \frac{1}{2n}|c|^2, \; n \in \mathbb{N}.$$

Indeed if we denote with $S_n f(x) = \inf_{y \in \mathbb{R}} \left\{ f(y) + \frac{n}{2}|x-y|^2 \right\}$ the Yosida approximation of a function f, then it is easy to check that $[S_n f]^*(x) = f^*(x) + \frac{1}{2n}|x|^2$, so that

$$h_n^*(c) = S_n(h^*)(c).$$

Being h_n^* the Yosida approximations of a *l.s.c.* convex function, they result to be Frechét differentiable with Lipschitz gradient, with Lipschitz constant $[(h_n^*)']_L \le n$. Moreover, as h_n is a decreasing sequence, F_n is then increasing, as required by Definiton 6.1. Hence the assumptions in Theorem 7.11 are satisfied for the problem of minimizing the functional

$$J_n(t,x,c) = J(t,x,c) + \frac{1}{2n}\int_t^T e^{-\rho s}|c(s)|^2 ds$$

in $L^2([t,T],\mathbb{R})$, and we easily derive as a consequence the following result.

Lemma 6.3. *Let*

$$W_n(t,x) \stackrel{def}{=} \inf_{c \in L^2([t,T],\mathbb{R})} J_n(t,x,c),$$

be the value functions of the approximating optimal control problem. Then W_n is convex in x and l.s.c. in x and t, and it is the unique weak solution of

$$\begin{cases} \partial_t v(t,x) + \langle \nabla v(t,x), G^* x \rangle - F_n(t, \nabla v(t,x)) + e^{-\rho t} g(x) = 0 \\ v(T,x) = \phi(x) \end{cases}$$

Moreover there exists $c_n^* \in L^2([t,T],\mathbb{R})$ optimal for the approximating problems, i.e. $W_n(t,x) = J_n(t,x,c_n^*)$.

To complete the proof we need to show that $W_n(t,x) \downarrow W(t,x)$.

Lemma 6.4. *The value function of (18) is given by*

$$W(t,x) = \lim_{n\to\infty} W_n(t,x) = \inf_n W_n(t,x).$$

Proof. By definition of J_n, for all t, x and n we have $J_n(t,x,c) \ge J_{n+1}(t,x,c)$ for all admissible controls c, so that

$$W_n(t,x) \ge W_{n+1}(t,x),$$

and $\{W_n(t,x)\}_n$ is a decreasing sequence. As a consequence, an ultraweak solution V of HJB exists, and it is given by

$$V(t,x) \stackrel{def}{=} \lim_{n\to\infty} W_n(t,x) = \inf_{n\in\mathbb{N}} W_n(t,x).$$

Next we show that a solution V built this way necessarily coincides with W. Note that

$$J(t,x,c) \leq J_n(t,x,c), \quad \forall c \in L^2([t,T],\mathbb{R}),$$

so that by taking the infimum and then passing to limits, we obtain

$$W(t,x) \leq V(t,x). \tag{19}$$

We then prove the reverse inequality. Let $\varepsilon > 0$ be arbitrarily fixed, and c_ε be an ε-optimal control for the problem, that is $W(t,x)+\varepsilon > J(t,x,c_\varepsilon)$. Note that, by passing to limits as $n \to +\infty$ in

$$V(t,x) \leq W_n(t,x) \leq J_n(t,x,c_\varepsilon)$$

one obtains

$$V(t,x) \leq J(t,x,c_\varepsilon) < W(t,x) + \varepsilon,$$

which implies, together with (19), the thesis.

Doing so we proved the lemma and Theorem 6.2.

Remark 6.5. *Note that we do not derive any uniqueness result for ultraweak solutions. If for instance one tries to get uniqueness by showing that any ultraweak solution of HJB is the value function of a certain control problem, some difficulties arise, due to the fact that, although $h_n^* \uparrow \mathcal{H}$ if and only if there exists some h such that $h_n \downarrow h$, in general $\mathcal{H}^* \neq h$ unless some minimax condition is satisfied, such as*

$$h = \inf_n \sup_r \{cr - h_n^*(r)\} = \sup_r \inf_n \{cr - h_n^*(r)\} = \mathcal{H}^*,$$

which is false in general.

7 Appendix

In this section we recall the abstract framework and the main results contained in [19] and [20], regarding strong and weak solutions of HJB.

In [19] and [20] we worked in an abstract setting on some state space denoted with V'. In that setting, if H is a separable Hilbert space, A_0 is the generator of a strongly continuous semigroup of operators on H, and V is the Hilbert space $D(A_0^*)$ endowed with the scalar product $(v|w)_V := (v|w)_H + (A_0^* v|A_0^* w)_H$, then we set V' equal to its dual space endowed with the operator norm. The semigroup generated by A_0 can be extended in a standard way to a semigroup $\{e^{As}\}_{s \geq 0}$ on the space V', with generator A, a proper extension of A_0.

Then we assume the state equation in V' is given by

$$\begin{cases} y'(s) = Ay(s) + Bc(s), & s \in [t,T] \\ y(t) = x \in V' \end{cases} \tag{20}$$

with control operator $B \in L(U,V')$ (although $B \notin L(U,H)$), where U is the control space and $c \in L^2([t,T],U)$ the control. Such equation may be readily expressed in mild form as

$$y(s) = e^{A(s-t)}x + \int_t^s e^{A(s-\sigma)}Bc(\sigma)d\sigma. \tag{21}$$

Remark 7.1. *The role of V' in the case of the delay equation here presented is played by the space $D(G)'$, and the role of A_0 by the operator G^*.*

Besides, we consider a target functional J_0, associated to the state equation, of type

$$J(t,x,c) = \int_t^T \left[g\left(s, y(s)\right) + h\left(s, c(s)\right) \right] ds + \varphi(y(\tau)) \tag{22}$$

with $h(t,\cdot)$ real, convex, *l.s.c.*, coercive, and $g(t,\cdot)$ and ν real, convex, and $C^1(V')$ (respectively, *l.s.c.* in V') in the x variable, as more precisely stated in the next sections. The problem is that of minimizing $J(t,x,\cdot)$ over the set of admissible controls $L^2([t,T];U)$.

Remark 7.2. *Indeed, in the applications, the target functional is rather of type*

$$J_0(t,x,c) = \int_t^T \left[\xi\left(s, y(s)\right) + \eta\left(s, c(s)\right) \right] ds + \nu(y(T))$$

with $\eta(t,\cdot)$ real, convex, l.s.c., coercive, and $\xi(t,\cdot)$ and ν real, convex, and $C^1(H)$ (respectively, l.s.c. in H) in the x variable, defined on H, but not necessarily on V'. Then we need to assume *that ξ and ν allow C^1 (respectively, l.s.c.) extensions $g(t,\cdot)$ and ϕ on the space V'. The existence of such extensions is of course a strong assumption, see [19] for details and comments upon this matter.*

Moreover, the value function is defined as

$$W(t,x) = \inf_{c \in L^2([t,T];U)} J(t,x,c), \tag{23}$$

Finally, we considered the following (backward) HJB equation associated to the problem set in $[0,T] \times V'$

$$\begin{cases} v_t(t,x) - \mathcal{H}(t, B^* \nabla v(t,x)) + \langle Ax | \nabla v(t,x) \rangle + g(t,x) = 0, \\ v(T,x) = \varphi(x), \end{cases} \tag{24}$$

for all t in $[0,T]$ and x in $D(A)$ (indeed for all x in V'), where

$$\mathcal{H}(t,c) = [h(t,\cdot)]^*(-c).$$

Note that $\mathcal{H}$ is well defined only for p in V, that is a proper subspace of H, to which $\nabla v(t,x)$ (the spatial gradient of v) belongs.

With such a problem in mind, we then investigate existence and uniqueness for the following forward HJB equation

$$\begin{cases} \phi_t(t,x) + F(t, \nabla \phi(t,x)) - \langle Ax, \nabla \phi(t,x) \rangle = g(T-t,x), \quad (t,x) \in [0,T] \times V' \\ \phi(0,x) = \varphi(x). \end{cases} \tag{25}$$

Note in fact that such a HJB is the forward version of (24) if we set

$$F(t,p) := \mathcal{H}(t, B^* p) = \sup_{c \in U} \left\{ \big(-Bc | p \big)_U - h(t,c) \right\}.$$

7.1 Regular data and strong solutions of HJB equations.

We first treat the case of regular data, from which the notion of strong solution originates.

Assumptions 7.3. *1. $A : D(A) \subset V' \to V'$ is the infinitesimal generator of a strongly continuous semigroup $\{e^{sA}\}_{s\geq 0}$ on V';*

2. *$B \in L(U, V')$;*
3. *there exists $\omega > 0$ such that $|e^{\tau A}x|_{V'} \leq M e^{\omega\tau}|x|_{V'}, \ \forall \tau \geq 0$;*
4. *$F \in \mathcal{Y}([0,T] \times V)$, $F(t,0) = 0$, $\sup_{t\in[0,T]}[F_p(t,\cdot)]_L < +\infty$;*
5. *$g \in \mathcal{Y}([0,T] \times V')$, $t \mapsto [g_x(t,\cdot)]_L \in L^1(0,T)$*
6. *$\varphi \in \Sigma_0(V')$;*
7. *$h(t,\cdot)$ is convex, lower semi–continuous, $\partial_c h(t,\cdot)$ is injective for all $t \in [0,T]$.*
8. *$\mathcal{H} \in \mathcal{Y}([0,T] \times U)$, $\mathcal{H}(t,0) = 0$, and $\sup_{t\in[0,T]}[\mathcal{H}_c(t,\cdot)]_L < +\infty$.*

Definition 7.4. *Let Assumptions 7.3 be satisfied. We say that $\phi \in C([0,T], C_2(V'))$ is a* strong *solution of (25) if there exists a family $\{\phi^\varepsilon\}_\varepsilon \subset C([0,T], C_2(V'))$ such that:*

(i) $\phi^\varepsilon(t,\cdot) \in C^1_{Lip}(V')$ and $\phi^\varepsilon(t,\cdot)$ is convex for all $t \in [0,T]$; $\phi^\varepsilon(0,x) = \varphi(x)$ for all $x \in V'$.

(ii) there exist constants $\Gamma_1, \Gamma_2 > 0$ such that

$$\sup_{t\in[0,T]} [\nabla\phi^\varepsilon(t)]_L \leq \Gamma_1, \ \ \sup_{t\in[0,T]} |\nabla\phi^\varepsilon(t,0)|_V \leq \Gamma_2, \ \forall \varepsilon > 0;$$

(iii) for all $x \in D(A)$, $t \mapsto \phi^\varepsilon(t,x)$ is continuously differentiable;

(iv) $\phi^\varepsilon \to \phi$, as $\varepsilon \to 0+$, in $C([0,T], C_2(V'))$;

(v) there exists $g_\varepsilon \in C([0,T]; C_2(V'))$ such that, for all $t \in [0,T]$ and $x \in D(A)$,

$$\phi^\varepsilon_t(t,x) - F(t, \nabla\phi^\varepsilon(t,x)) + \langle Ax, \nabla\phi^\varepsilon(t,x)\rangle_{V'} = g_\varepsilon(T-t,x)$$

with $g_\varepsilon(t,x) \to g_0(t,x)$, and $\int_0^T |g_\varepsilon(s) - g_0(s)|_{C_2} ds \to 0$, as $\varepsilon \to 0+$.

The main result contained in [19] is the following.

Theorem 7.5. *Let Assumptions 7.3 be satisfied. There exists a unique strong solution ϕ of (25) in the class $C([0,T], C_2(V'))$ with the following properties:*

(i) for all $x \in D(A)$, $\phi(\cdot,x)$ is Lipschitz continuous;

(ii) $\phi(t,\cdot) \in \Sigma_0(V')$, for all $t \in [0,T]$.

Regarding applications to the optimal control problem, in [?] we were able to prove what follows.

Theorem 7.6. *Let Assumptions 7.3 be satisfied, with $F(t,p) := \mathcal{H}(t, B^*p)$. Let W be the value function of the control problem, and let ϕ be the strong solution of (25) described in Theorem 7.5. Then*

$$W(t,x) = \phi(T-t,x), \ \forall t \in [0,T], \ \forall x \in V',$$

that is, the value function W of the optimal control problem is the unique strong solution of the backward HJB equation (24).

7.2 Semicontinuous data and weak solutions of HJB equations.

We then treat the case of merely semicontinuous data, from which the notion of *weak* solution originates.

Assumptions 7.7. *If K is a convex closed subset of V', we define*

$$\Sigma_K \equiv \Sigma_K(V') := \{\phi : V' \to (-\infty, +\infty] \ : \ \phi \text{ is convex and } l.s.c., \ K \subset D(\phi)\}$$

where $D(\phi) = \{x \in V' \ : \ \phi(x) < +\infty\}$, and assume:

1. *$C : D(C) \subset V' \to V'$ is the infinitesimal generator of a strongly continuous semigroup $\{e^{sA}\}_{s\geq 0}$ on V';*
2. *$B \in L(U, V')$;*
3. *there exists $\omega > 0$ such that $|e^{sC}x|_{V'} \leq e^{\omega s}|x|_{V'}, \ \forall s \geq 0$;*
4. *$F \in \mathcal{Y}([0,T] \times V)$, $F(t,0) = 0$, $\sup_{t\in[0,T]}[F_p(t,\cdot)]_L < +\infty$;*
5. *$g(t,\cdot) \in \Sigma_K(V')$, for all $t \in [0,T]$; $g(\cdot, x)$ l.s.c. and $L^1(0,T)$ for all $x \in V'$;*
6. *$\varphi \in \Sigma_K(V')$;*
7. *$h(t,\cdot)$ is convex, lower semi–continuous, $\partial_c h(t,\cdot)$ is injective for all $t \in [0,T]$; moreover $h(t,c) \geq a(t)|c|_U^2 + b(t)$, with $a(t) \geq A(T) > 0$, $b \in L^1(0,T;\mathbb{R})$;*
8. *$\mathcal{H} \in \mathcal{Y}([0,T] \times U)$, $\mathcal{H}(t,0) = 0$, and $\sup_{t\in[0,T]}[\mathcal{H}_c(t,\cdot)]_L < +\infty$.*

Definition 7.8. *Let $K \subset V'$ be a closed convex set, and let $\varphi \in \Sigma_K$ and $g(t,\cdot) \in \Sigma_K$ for all t in $[0,T]$. Then $\phi : [0,T] \times V' \to (-\infty, +\infty]$ is a* weak *solution of (HJB) if:*
(i) $\phi(t,\cdot) \in \Sigma_K, \ \forall t \in [0,T]$;
(ii) there exist sequences $\{\varphi_n\}_n \subset \Sigma_0$, and $\{g_n\} \subset \mathcal{Y}([0,T] \times V')$, such that

$$\varphi_n(x) \uparrow \varphi(x), \ g_n(t,x) \uparrow g(t,x), \ \forall x \in V', \ \forall t \in [0,T], \ \text{as } n \to +\infty,$$

and moreover, if ϕ_n is the unique strong solution of

$$\begin{cases} \phi_t(t,x) + F(t, \nabla\phi(t,x)) - \langle Ax, \nabla\phi(t,x)\rangle_{V'} = g_n(t,x) & (t,x) \in [0,T] \times V' \\ \phi(0,x) = \varphi_n(x) \end{cases}$$

in $C([0,T], C_2(V'))$, then

$$\phi_n(t,x) \uparrow \phi(t,x), \ \forall (t,x) \in [0,T] \times V'.$$

Remark 7.9. *Since strong solution were proved in [19] to be Lipschtiz with respect to the time variable and C^1 with respect to the space variable, and the weak solution ϕ is a sup–envelop of strong solutions ϕ_n, then ϕ is lower semicontinuous in $[0,T] \times V'$. For the same reason ϕ_n convex in the x variable implies that ϕ is convex in x as well.*

Remark 7.10. *Note that the role of the convex set K is played in the first example by the set*

$$K \stackrel{def}{=} cl_{V'}(\{(x_0, x_1) \ : \ x_0 \geq 0\})$$

Theorem 7.11. *Let Assumptions 7.7 be satisfied. Let also g and h be of the following type*

$$g(t,x) = e^{-\rho t} g_0(x), \quad h(t,c) = e^{-\rho t} h_0(x)$$

. Then the following properties are equivalent:

(i) there exists a unique weak solution of (25);

(ii) At each $(t,x) \in [0,T] \times K$ there exists an admissible control.

Moreover if (i) or (ii) holds, there exists an optimal pair (c^*, y^*) and

$$\phi(T-t,x) = J(t,x,c^*).$$

References

[1] V. Barbu and G. Da Prato. *Hamilton-Jacobi equations in Hilbert spaces*, volume 86 of *Research Notes in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1983.

[2] V. Barbu and G. Da Prato. Hamilton-Jacobi equations in Hilbert spaces: variational and semigroup approach. *Ann. Mat. Pura Appl. (4)*, 142:303–349 (1986), 1985.

[3] V. Barbu, G. Da Prato, and C. Popa. Existence and uniqueness of the dynamic programming equation in Hilbert space. *Nonlinear Anal.*, 7(3):283–299, 1983.

[4] E. Barucci and F. Gozzi. Optimal advertising with a continuum of goods. *Ann. Oper. Res.*, 88:15–29, 1999. Optimal control and differential games (Vienna, 1997).

[5] A. Bensoussan, G. Da Prato, M. C. Delfour, and S. K. Mitter. *Representation and control of Infinite dimensional system.* Birkhäuser Boston, 1992.

[6] R. Boucekkine, D. de la Croix, and O. Licandro. Modelling vintage structures with *ddes*: principles and applications. *Math. Popul. Stud.*, 11(3-4):151–179, 2004.

[7] R. Boucekkine, O. Licandro, L. A. Puch, and F. del Rio. Vintage capital and the dynamics of the AK model. *J. Econom. Theory*, 120(1):39–72, 2005.

[8] M. Crandall and P.L. Lions. User's guide to viscosity solution of hamilton-Jacobi equations. *Bull. Amer. Math. Soc. (4)*, 142:303–349 (1986), 1985.

[9] M. C. Delfour. Status of the state space theory of linear hereditary differential systems with delays in state and control variables. In *Analysis and optimization of systems (Proc. Fourth Internat. Conf., Versailles, 1980)*, volume 28 of *Lecture Notes in Control and Information Sci.*, pages 83–96. Springer, Berlin, 1980.

[10] M. C. Delfour. Linear optimal control of systems with state and control variable delays. *Automatica J. IFAC*, 20(1):69–77, 1984.

[11] M. C. Delfour. The linear quadratic optimal control problem with delays in the state and control variables: a state space approach. *SIAM J. Contr. Optimiz.*, 24(1):835–883, 1986.

[12] M. C. Delfour and A. Manitius. Control systems with delays: areas of applications and present status of the linear theory. In *New trends in systems analysis (Proc. Internat. Sympos., Versailles, 1976)*, pages 420–437. Lecture Notes in Control and Informat. Sci., Vol. 2. Springer, Berlin, 1977.

[13] M. C. Delfour, C. McCalla, and S. K. Mitter. Stability and the infinite-time quadratic cost problem for linear hereditary differential systems. *SIAM J. Control*, 13:48–88, 1975.

[14] M. C. Delfour and S. K. Mitter. Controllability and observability for infinite-dimensional systems. *SIAM J. Control*, 10:329–333, 1972.

[15] M. C. Delfour and S. K. Mitter. Hereditary differential systems with constant delays. II. A class of affine systems and the adjoint problem. *J. Differential Equations*, 18:18–28, 1975.

[16] O. Diekmann, S. A. van Gils, S. M. Verduyn Lunel, and H. Walther. *Delay equations*, volume 110 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1995. Functional, complex, and nonlinear analysis.

[17] G. Fabbri and F. Gozzi. Vintage capital in the ak growth model: a dynamic programming approach. march 2006.

[18] S. Faggian. Regular solutions of first-order Hamilton-Jacobi equations for boundary control problems and applications to economics. *Appl. Math. Optim.*, 51(2):123–162, 2005.

[19] S. Faggian. Regular solutions of first-order Hamilton-Jacobi equations for boundary control problems and applications to economics. *Appl. Math. Optim.*, 51(2):123–162, 2005.

[20] S. Faggian. Infinite dimensional hamilton–jacobi equations and applications to boundary control problems with state constraints. march 2006.

[21] S. Faggian and F. Gozzi. On the dynamic programming approach for optimal control problems of pde's with age structure. *Mathematics for Population Studies*, 11 (3-4):233–270, 2004.

[22] G. Feichtinger, R. Hartl, P. Kort, and V. Veliov. Anticipation effects of technological progress on capital accumulation: a vintage capital approach. *Journal of Economic Theory*, 20:645–668, 2006.

[23] W.H. Fleming and R.W. Rishel. *Deterministic and stochastic optimal control.* Springer-Verlag, Berlin, 1975. Applications of Mathematics, No. 1.

[24] F. Gozzi and C. Marinelli. Optimal advertising under uncertainty with memory and lags. mimeo, 2004. to appear in the proceedings of the conference "SPDE and Applications" 2004.

[25] F. Gozzi, C. Marinelli, and Savin S. Optimal advertising under uncertainty with carryover effects. march 2006.

[26] A. Ichikawa. Evolution equations, quadratic control, and filtering with delay. In *Analyse et contrôle de systèmes (Papers, IRIA Sem., Rocquencourt, 1977)*, pages 117–126. IRIA, Rocquencourt, 1977.

[27] A. Ichikawa. Quadratic control of evolution equation with delay in control. *SIAM journal of control and optimization*, 20:645–668, 1982.

[28] V. B. Kolmanovskiĭ and L. E. Shaĭkhet. *Control of systems with aftereffect*, volume 157 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1996. Translated from *Control of systems with aftereffect* (Russian) [“Nauka”, Moscow, 1992; MR1185708 (93i:49001)] by Victor Kotov.

[29] X. Li and J. Yong. *Optimal control theory for infinite dimensional system.* Birkhauser, 1995.

[30] R. B. Vinter and R. H. Kwong. The infinite time quadratic control problem for linear system with state control delays: An evolution equation approch. *SIAM journal of control and optimization*, 19:139–153, 1981.